02-Discrete-Time Signals and Systems

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BASIC DEFINITIONS

Signals may be classified into four categories depending on the characteristics of **the time-variable** and **values** they can take:

- continuous-time signals (analogue signals),
- discrete-time signals,
- continuous-valued signals,
- discrete-valued signals.

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CONTINUOUS-TIME (ANALOGUE) SIGNALS

:

Time: defined for every value of time $t \in R$,
Descriptions: functions of a continuous variable t: $f(t)$,
Notes: they take on values in the continuous interval $f(t) \in (-a,b)$ *for* $a,b \to \infty$.

Note: $f(t) \in C$

$$f(t) = \sigma + j\omega$$

$$\sigma \in (-a,b) \, and \, \omega \in (-a,b)$$

$$a,b \to \infty$$

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DISCRETE-TIME SIGNALS

:

Time: defined **only** at discrete values of time: $t = nT$,
Descriptions: sequences of real or complex numbers $f(nT) = f(n)$,
Note A.: they take on values in the continuous interval $f(n) \in (-a,b)$ *for* $a,b \to \infty$,
Note B.: sampling of analogue signals:

- sampling interval, period: T,
- sampling rate: *number of samples per second,*
- sampling frequency (Hz): $f_S = 1/T$.

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CONTINUOUS-VALUED SIGNALS

:

Time: they are defined **for every value of time** or only at discrete values of time,

Value: they can take on **all possible values** on finite or infinite range,

Descriptions: functions of a continuous variable or sequences of numbers.

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Discrete-valued signals:

Time: they are defined for **every value of time** or only at discrete values of time,

Value: they can take on values from **a finite set** of possible values,

Descriptions: functions of a continuous variable or sequences of numbers.

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DIGITAL FILTER THEORY:

Discrete-time signals:

Definition and descriptions: defined only at **discrete** values of time and they can take **all** possible values on finite or infinite range (**sequences** of real or complex numbers: $f(n)$),

Note: sampling process, constant sampling period.

Digital signals:

Definition and descriptions: **discrete-time and discrete-valued signals** (i.e. discrete -time signals taking on values from a finite set of possible values),

Note: sampling, quatizing and coding process i.e. process of analogue-to-digital conversion.

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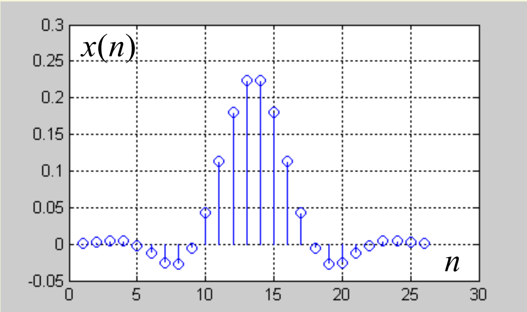
DISCRETE-TIME SIGNAL REPRESENTATIONS

A. Functional representation:

$$x(n) = \begin{cases} 1 & for \quad n = 1,3 \\ 6 & for \quad n = 0,7 \\ 0 & elsewhere \end{cases} \qquad y(n) = \begin{cases} 0 & for \quad n < 0 \\ 0,6^n & for \quad n = 0,1,\ldots,102 \\ 1 & \quad n > 102 \end{cases}$$

B. Graphical representation



DISCRETE-TIME SIGNAL

- In Matlab, a finite-duration sequence representation requires two vectors, and each for x and *n*.
  - Example:
  - Question: *whether or not an arbitrary infinite-duration sequence can be represented in MATLAB?*

DISCRETE-TIME SIGNAL REPRESENTATIONS

C. Tabular representation:

D. Sequence representation:

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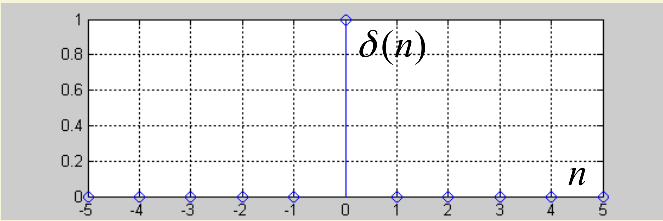
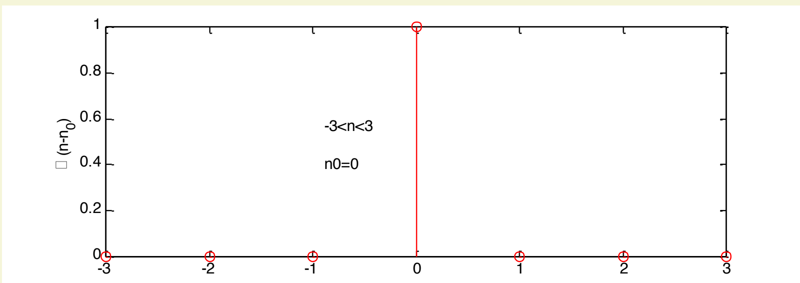
ELEMENTARY DISCRETE-TIME SIGNALS

A. Unit sample sequence (unit sample, unit impulse, unit impulse signal)



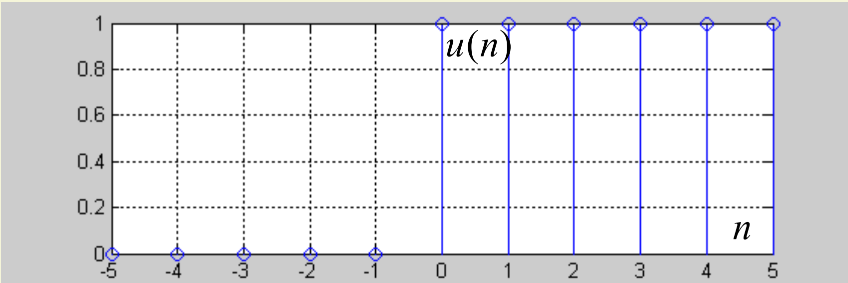
FUNCTION [X,N]=IMPSEQ(N_0,N_1,N_2)

- A: n=[n1:n2];
  - x = zeros(1,n2-n1+1); x(n0-n1+1)=1;
- B: n=[n1:n2]; x = [(n-n0)==0]; stem(n,x,'ro');



ELEMENTARY DISCRETE-TIME SIGNALS

B. Unit step signal (unit step, Heaviside step sequence)



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UNIT STEP SEQUENCE

$$u(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} = \left\{ \cdots, 0, 0, \underset{\uparrow}{1}, 1, 1, \cdots \right\}$$

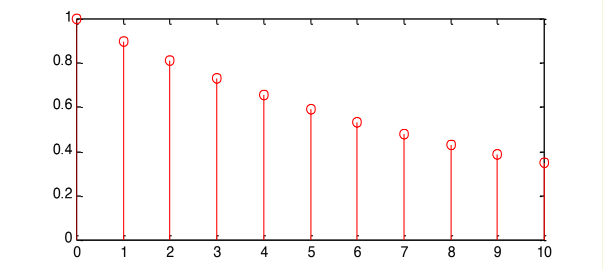
A: n=[n1:n2]; x=zeros(1,n2-n2+1); x(n0-n1+1:end)=1;

B: n=[n1:n2]; x=[(n-n0)>=0];

3. REAL-VALUED EXPONENTIAL SEQUENCE

For Example: $x(n) = (0.9)^n, \ 0 \le n \le 10$

n=[0:10]; x=(0.9).^n; stem(n,x,’ ro’)



ELEMENTARY DISCRETE-TIME SIGNALS

C. Complex-valued exponential signal

(complex sinusoidal sequence, complex phasor)

$$x(n) = e^{j\omega nT}, \ |x(n)| = 1, \ \arg[x(n)] = \omega nT = 2\pi f.nT = \frac{2\pi f.n}{f_S}$$

where

$$\omega \in R, \ n \in N, \ j = \sqrt{-1} \ \ is \ \ imaginary \ \ unit$$

and

T is sampling period and f_S is sampling frequency.

For Example: n=[0:10]; x=exp((2+3j)*n);

DISCRETE-TIME SYSTEMS

A discrete-time system is a device or algorithm that operates on a discrete-time signal called ***the input*** or ***excitation*** (e.g. ***x(n)***), according ***to some rule*** (e.g. ***H[.]***) to produce another discrete-time signal called *the* ***output*** or ***response*** (e.g. ***y(n)***).

This expression denotes also **the transformation *H[.]* (also called operator or mapping) or processing performed by the system on *x(n)* to produce *y(n)*.**

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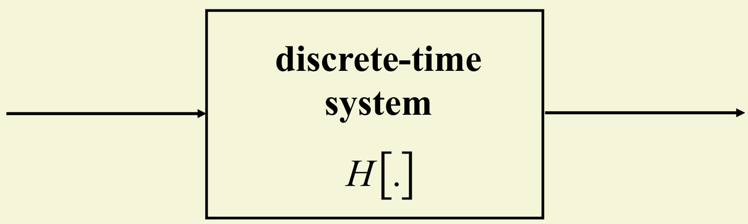
DISCRETE-TIME SYSTEMS

Input-Output Model of Discrete-Time System

(input-output relationship description)



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CLASSIFICATION. STATIC VS. DYNAMIC SYSTEMS

A discrete-time system is called ***static*** or ***memoryless*** if its output at any time instant n depends on the input sample at the same time, but not on the past or future samples of the input. In the other case, the system is said to be ***dynamic*** or to have ***memory***.

If the output of a system at time n is completly determined by the input samples in the interval from n-N to n ($N \geq 0$), the system is said to have memory of ***duration N***.

If $N = 0$, the system is ***static*** or ***memoryless***.

If $0 < N < \infty$, the system is said to have ***finite memory***.

If $N \to \infty$, the system is said to have ***infinite memory***.

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Examples:

The static (memoryless) systems:

$$y(n) = nx(n) + bx^3(n)$$

The dynamic systems with finite memory:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

The dynamic system with infinite memory:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

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.TIME-INVARIANT VS. TIME-VARIABLE

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A discrete-time system is called ***time-invariant*** if its input-output characteristics do not change with time. In the other case, the system is called ***time-variable***.

Definition. A relaxed system $H[.]$ is ***time-*** or ***shift-invariant*** if only if

$$y(n) \equiv H\left[x(n)\right] \qquad x(n) \xrightarrow{H} y(n)$$

implies that

$$y(n-k) \equiv H\left[x(n-k)\right] \qquad x(n-k) \xrightarrow{H} y(n-k)$$

for ***every input signal*** $x(n)$ and ***every time shift*** k .

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Examples:

The time-invariant systems:

$$y(n) = x(n) + bx^3(n)$$

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

The time-variable systems:

$$y(n) = nx(n) + bx^3(n-1)$$

$$y(n) = \sum_{k=0}^{N} h^{N-n}(k)x(n-k)$$

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LINEAR VS. NON-LINEAR SYSTEMS

A discrete-time system is called ***linear*** if only if it satisfies the ***linear superposition principle***. In the other case, the system is called ***non-linear***.

Definition. A relaxed system $H[.]$ is ***linear*** if only if

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$, and any arbitrary constants a_1 and a_2.

The multiplicative (scaling) property of a linear system:

The additivity property of a linear system:

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Examples:

The linear systems:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) \qquad y(n) = x(n^2) + bx(n-k)$$

The non-linear systems:

$$y(n) = nx(n) + bx^3(n-1) \quad y(n) = \sum_{k=0}^{N} h(k)x(n-k)x(n-k+1)$$

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CAUSAL VS. NON-CAUSAL

Definition. A system is said to be ***causal*** if the output of the system at any time *n* (i.e., *y(n)*) depends only on present and past inputs (i.e., *x(n), x(n-1), x(n-2), …*). In mathematical terms, the output of a ***causal*** system satisfies an equation of the form

where $F[.]$ is some arbitrary function. If a system does not satisfy this definition, it is called ***non-causal***.

Examples:

The causal system:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) \qquad y(n) = x^2(n) + bx(n-k)$$

The non-causal system:

$$y(n) = nx(n+1) + bx^3(n-1) \qquad y(n) = \sum_{k=-10}^{10} h(k)x(n-k)$$

STABLE VS. UNSTABLE

An arbitrary relaxed system is said to be ***bounded input - bounded output (BIBO) stable*** if and only if every bounded input produces the bounded output. It means, that there exist some finite numbers say M_x and M_y, such that

for all n. If for some bounded input sequence $x(n)$, the output $y(n)$ is unbounded (infinite), the system is classified as ***unstable***.

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Examples:

The stable systems:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) \qquad y(n) = x(n^2) + 3x(n-k)$$

The unstable system:

$$y(n) = 3^n x^3(n-1)$$

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RECURSIVE VS. NON-RECURSIVE

A system whose output *y(n)* at time *n* depends on any number of the past outputs values (e.g. *y(n-1), y(n-2), …*), is called a ***recursive system***. Then, the output of a causal recursive system can be expressed in general as

where *F[.]* is some arbitrary function. In contrast, if *y(n)* at time *n* depends only on the present and past inputs

then such a system is called ***nonrecursive***.

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Examples:

The nonrecursive system:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

The recursive system:

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{N} a(k)y(n-k)$$

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SYSTEMS (LTI SYSTEMS)

TIME-DOMAIN REPRESENTATION

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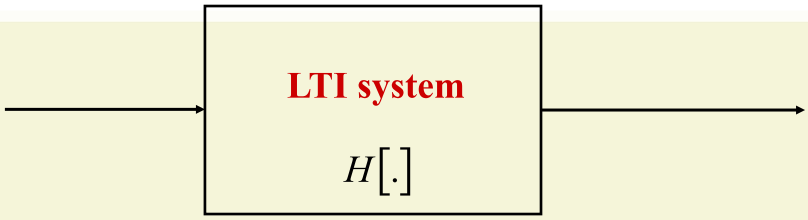
IMPULSE RESPONSE AND CONVOLUTION



LTI system description by **convolution** (convolution sum):

Viewed mathematically, the convolution operation satisfies the commutative law.

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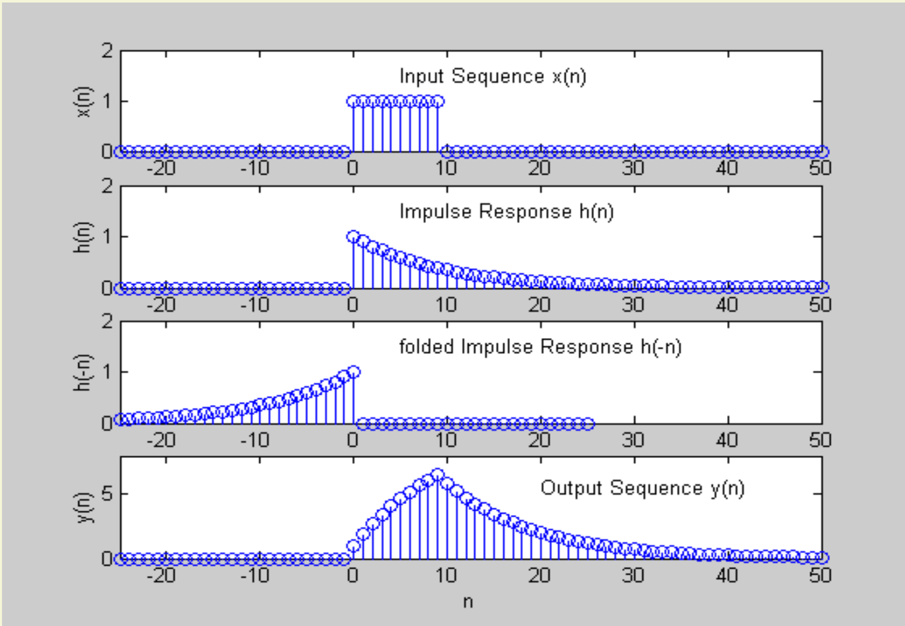
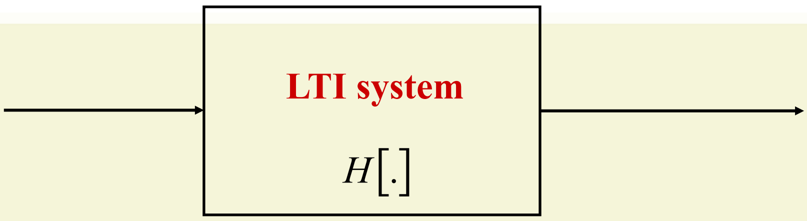


STEP RESPONSE



These expressions relate the impulse response to the step response of the system.

CLASSIFICATION OF LTI SYSTEMS. CAUSAL LTI SYSTEMS

A relaxed LTI system is ***causal*** if and only if its impulse response is zero for negative values of n , i.e.

Then, the two equivalent forms of the convolution formula can be obtained for the causal LTI system:

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STABLE LTI SYSTEMS

A LTI system is ***stable*** if its impulse response is absolutely summable, i.e.

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FINITE IMPULSE RESPONSE (FIR) & INFINITE IMPULSE RESPONSE (IIR)

Causal **FIR** LTI systems:

IIR LTI systems:

DIGITAL FILTER

- Discrete-time LTI systems are also called digital filter.
- Classification
  - FIR filter & IIR filter
- FIR filter
  - Finite-duration impulse response filter
  - Causal FIR filter
  - $h(0)=b_0,\ldots,h(M)=b_M$ $\quad y(n)=\sum_{m=0}^{M} b_m x(n-m)$
  - Nonrecursive or moving average (MA) filter
  - Difference equation coefficients, $\{b_m\}$ and $\{a_0=1\}$
  - Implementation in Matlab: Conv(x,h); filter(b,1,x)

IIR FILTER

- Infinite-duration impulse response filter
- Difference equation

- Recursive filter, *in which the output y(n) is recursively computed from its previously computed values*
- Autoregressive (AR) filter

ARMA FILTER

- Generalized IIR filter

$$y(n)=\sum_{m=0}^{M} b_m x(n-m)-\sum_{k=1}^{N} a_k y(n-k),\ n\geq 0$$

- It has two parts: MA part and AR part
- Autoregressive moving average filter, ARMA
- Solution
  - filter(b,a,x); %{b_m}, {a_k}

RECURSIVE AND NONRECURSIVE

Causal ***nonrecursive*** LTI:

Causal ***recursive*** LTI:

LTI systems:

characterized by ***constant-coefficient difference equations***

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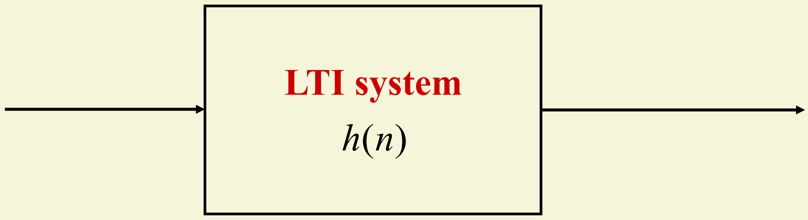
DIFFERENCE EQUATION

- An LTI discrete system can also be described by a *linear constant coefficient difference equation* of the form
- If a_N ~= 0, then the difference equation is of order N
- It describes a *recursive approach* for computing the current output,given the input values and previously computed output values.

FREQUENCY-DOMAIN



LTI system output:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega(n-k)} =$$

$$= \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}e^{j\omega n} = e^{j\omega n}\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

$$y(n) = e^{j\omega n}H(e^{j\omega})$$

Frequency response:

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\phi(\omega)}$$

$$H(e^{j\omega}) = \mathrm{Re}\left[H(e^{j\omega})\right] + j\,\mathrm{Im}\left[H(e^{j\omega})\right]$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)\cos\omega k + j\left[-\sum_{k=-\infty}^{\infty} h(k)\sin\omega k\right]$$

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Magnitude response:

Phase response:

Group delay function:

$$\tau(\omega) = -\frac{d\phi(\omega)}{d\omega}$$

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IMPULSE RESPONSE VS FREQUENCY RESPONSE

The important property of ***the frequency response***

is fact that this function ***is periodic with period*** 2π.

In fact, we may view the previous expression as the exponential Fourier series expansion for $H(e^{j\omega})$, with *h(k)* as the Fourier series coefficients. Consequently, the unit impulse response *h(k)* is related to $H(e^{j\omega})$ through the integral expression

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SYMMETRY PROPERTIES

For LTI systems with real-valued impulse response, the magnitude response, phase responses, the real component of and the imaginary component of $H(e^{j\omega})$ possess these symmetry properties:

The real component: ***even function*** of ω periodic with period 2π

The imaginary component: ***odd function*** of ω periodic with period 2π

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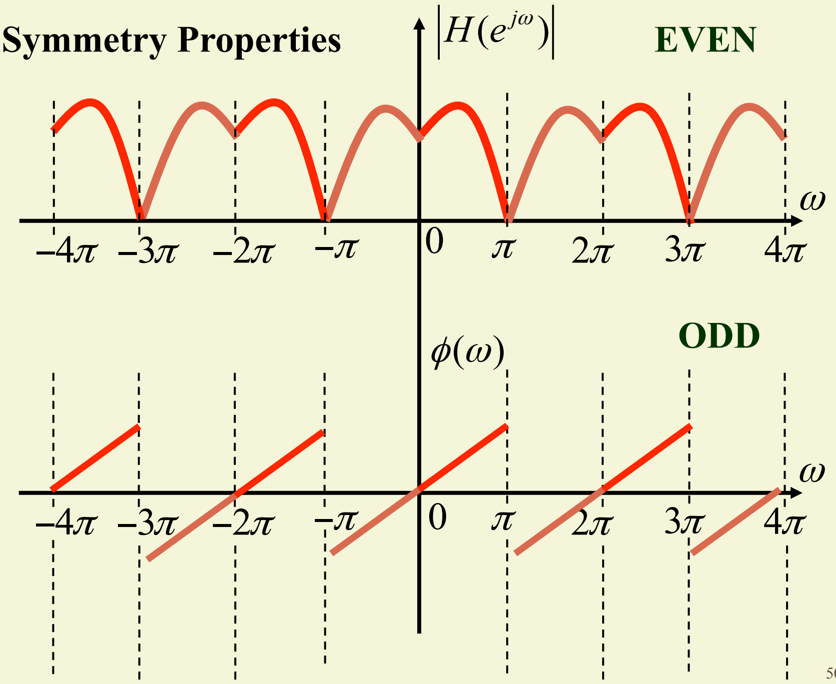
The magnitude response: even function of ω periodic with period 2π

The phase response: odd function of ω periodic with period 2π

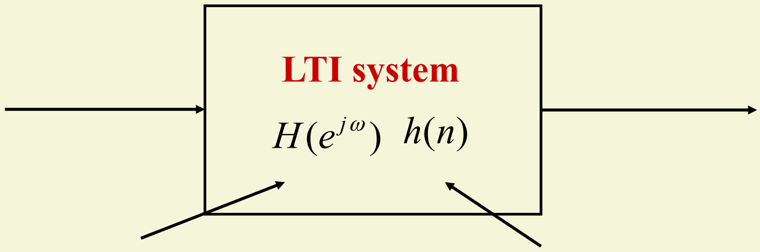
Consequence:

If we known $\left|H(e^{j\omega})\right|$ and $\phi(\omega)$ for $0 \leq \omega \leq \pi,$ we can describe these functions (i.e. also $H(e^{j\omega})$) for all values of ω .

Symmetry Properties



FOURIER TRANSFORM AND FREQUENCY-DOMAIN



The input signal *x(n)* and the spectrum of *x(n)*:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k} \qquad x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

The output signal *y(n)* and the spectrum of *y(n)*:

$$Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} y(k)e^{-j\omega k} \qquad y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega n}d\omega$$

The impulse response *h(n)* and the spectrum of *h(n)*:

Frequency-domain description of LTI system:

NORMALIZED FREQUENCY

It is often desirable to express the frequency response of an LTI system in terms of units of frequency that involve sampling interval *T*. In this case, the expressions:

are modified to the form:

$H(e^{j\omega T})$ is periodic with period $2\pi / T = 2\pi F$, where F is sampling frequency.

Solution: **normalized frequency approach**: $F/2 \rightarrow \pi$

Example:

$F = 100kHz \quad F/2 = 50kHz \quad 50kHz \rightarrow \pi$

$$f_1 = 20kHz \qquad \omega_1 = \frac{20x10^3}{50x10^3}\pi = \frac{2\pi}{5} = 0.4\pi$$

$$f_2 = 25kHz \qquad \omega_2 = \frac{25x10^3}{50x10^3}\pi = \frac{\pi}{2} = 0.5\pi$$

TRANSFORM-DOMAIN REPRESENTATION

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Z -TRANSFORM

Definition: The Z – transform of a discrete-time signal $x(n)$ is defined as the power series:

where z is a complex variable. The above given relations are sometimes called **the direct *Z* - transform** because they transform the time-domain signal $x(n)$ into its complex-plane representation $X(z)$.

Since Z – transform is an infinite power series, it exists only for those values of z for which this series converges. The **region of convergence** of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value.

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The procedure for transforming from z – domain to the time-domain is called **the inverse *Z* – transform**. It can be shown that the inverse *Z* – transform is given by

where *C* denotes the closed contour in the region of convergence of *X(z)* that encircles the origin.

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TRANSFER FUNCTION

The LTI system can be described by means of **a constant coefficient linear difference equation** as follows

Application of the *Z*-transform to this equation under zero initial conditions leads to the notion of **a transfer function**.

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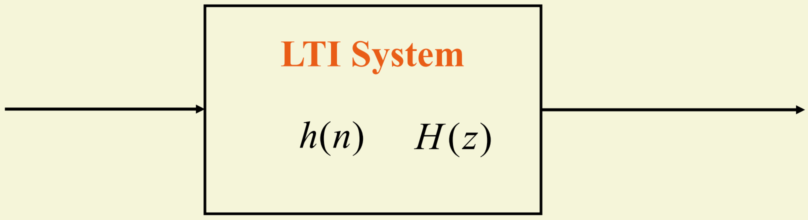

$$H(z) = Z\left[h(n)\right]$$

Transfer function: the ratio of the Z - transform of the output signal and the Z - transform of the input signal of the LTI system:

LTI system: the Z-transform of the constant coefficient linear difference equation under zero initial conditions:

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{M} a(k)y(n-k)$$

$$Y(z) = \sum_{k=0}^{N} b(k)z^{-k}X(z) - \sum_{k=1}^{M} a(k)z^{-k}Y(z)$$

The transfer function of the LTI system:

H(z): may be viewed as a rational function of a complex variable z (z^{-1}).

POLES AND ZEROS

Let us assume that $H(z)$ has been expressed in its irreducible or so-called factorized form:

Zeros of $H(z)$: the set $\{z_k\}$ of z-plane for which $H(z_k)=0$

Poles of $H(z)$: the set $\{p_k\}$ of z -plane for which $H(p_k) \to \infty$

Pole-zero plot: the plot of **the zeros** and **the poles** of $H(z)$ in the z-plane represents a strong tool for LTI system description.

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Example: the 4-th order Butterworth low-pass filter, cut off frequency $\omega_1 = \pi/3$.

b =[0.0186 0.0743 0.1114 0.0743 0.0186]

a =[1.0000 -1.5704 1.2756 -0.4844 0.0762]

z_1= -1.0002, z_2= -1.0000 + 0.0002j

z_3= -1.0000 - 0.0002j, z_4= -0.9998

p_1= 0.4488 + 0.5707j, p_2= 0.4488 - 0.5707j

p_3= 0.3364 + 0.1772j, p_4= 0.3364 - 0.1772j

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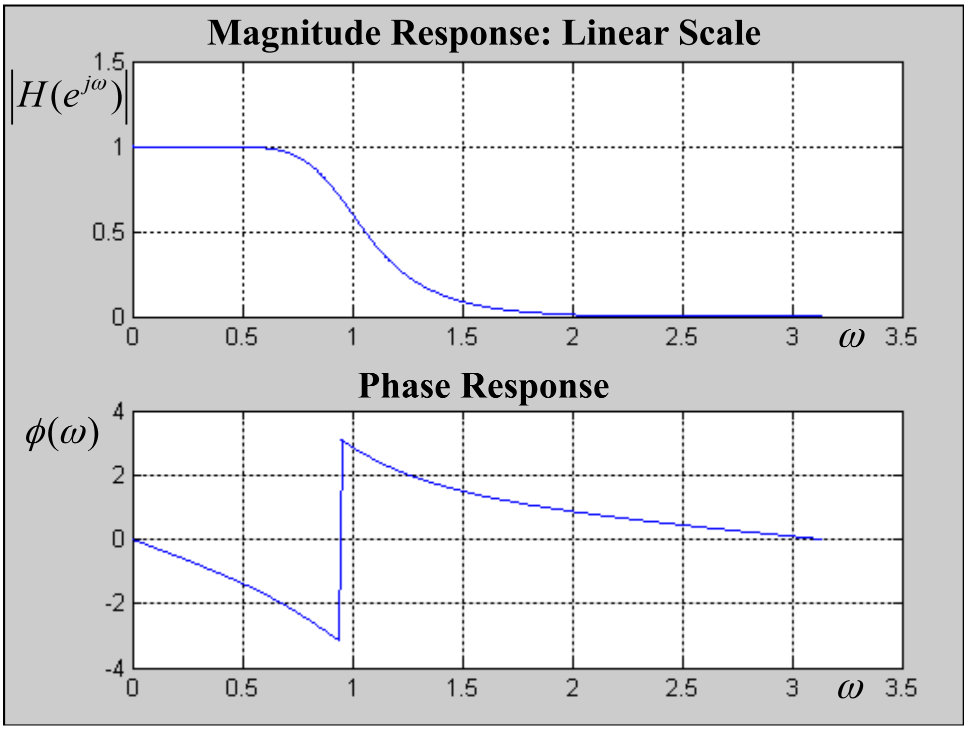
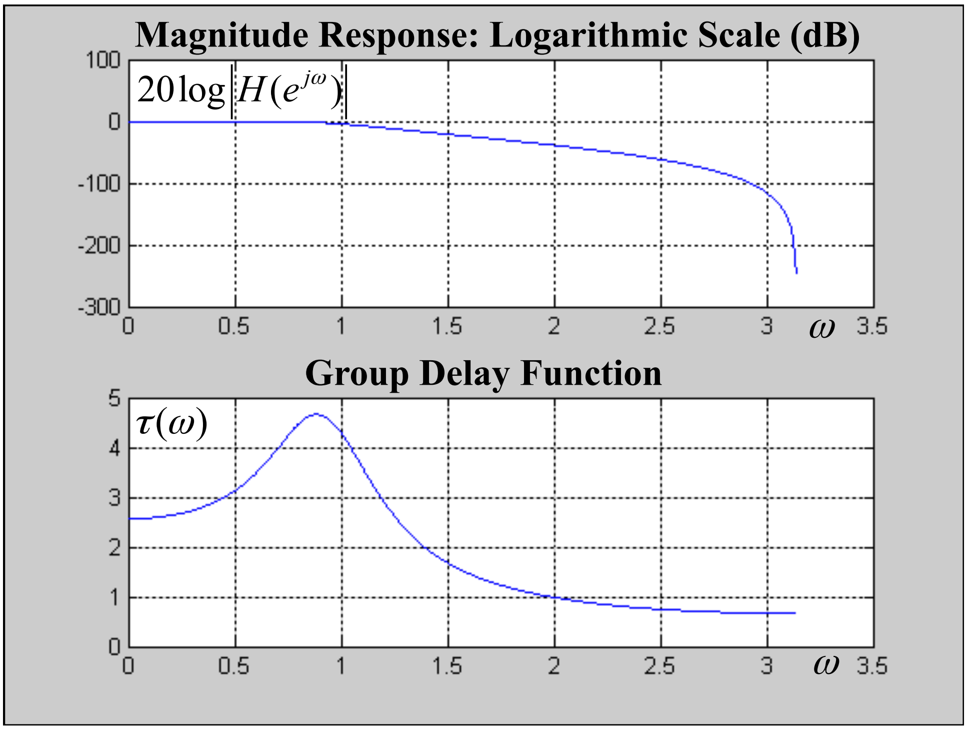
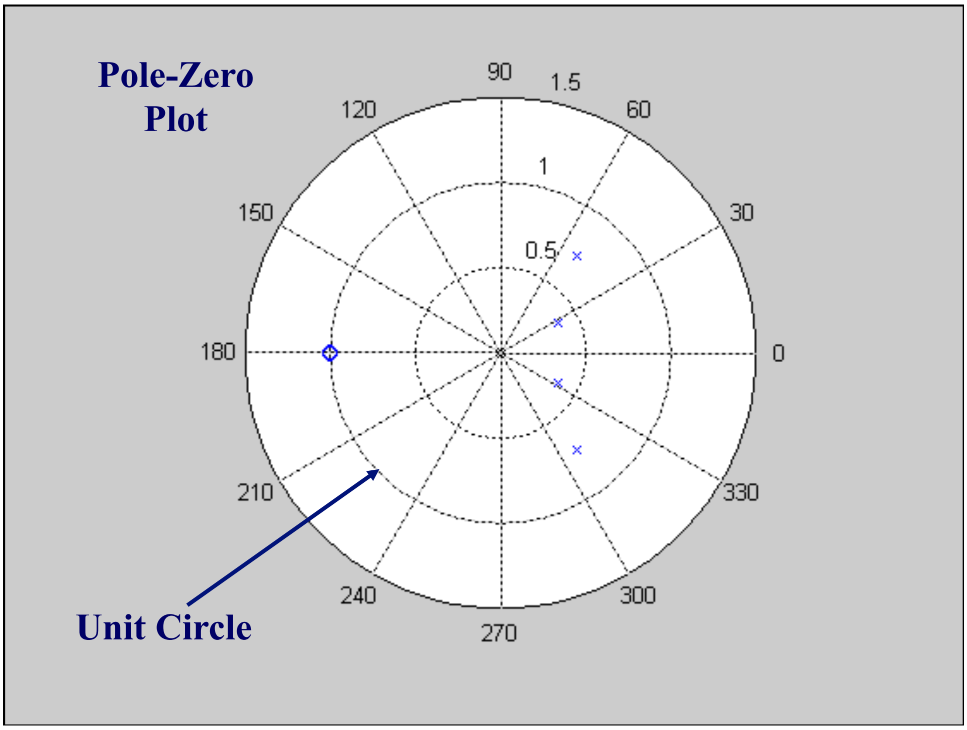
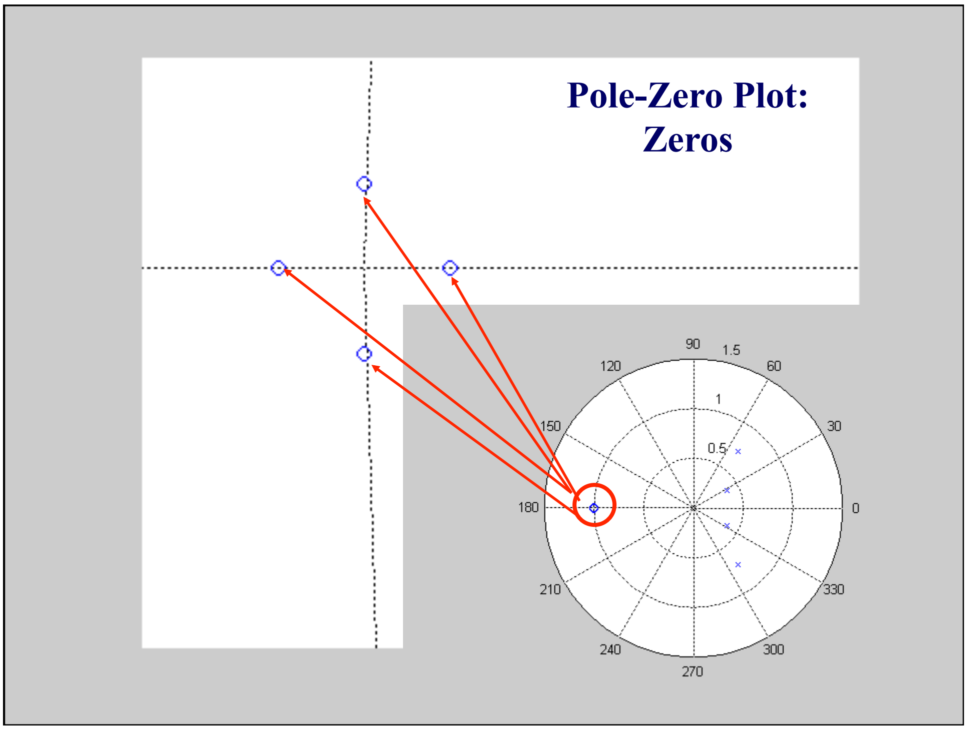

Magnitude Response: Linear Scale
$|H(e^{j\omega})|$
1.5
1
0.5
0
0 0.5 1 1.5 2 2.5 3 ω 3.5
Phase Response
$\phi(\omega)$
4
2
0
-2
-4
0 0.5 1 1.5 2 2.5 3 ω 3.5


Magnitude Response: Logarithmic Scale (dB)
$20\log|H(e^{j\omega})|$
100
0
-100
-200
-300
0 0.5 1 1.5 2 2.5 3 ω 3.5
Group Delay Function
$\tau(\omega)$
5
4
3
2
1
0
0 0.5 1 1.5 2 2.5 3 ω 3.5


Pole-Zero
Plot
90
1.5
120
60
1
150
30
0.5
180
0
210
330
240
300
270
Unit Circle


Pole-Zero Plot:
Zeros
90
1.5
120
60
1
150
30
0.5
180
0
210
330
240
300
270

1.4.4. Transfer Function and Stability of LTI Systems

Condition: LTI system is BIBO stable if and only if the unit circle falls within the region of convergence of the power series expansion for its transfer function. In the case when the transfer function characterizes a causal LTI system, the stability condition is equivalent to the requirement that **the transfer function *H(z)* has all of its poles inside the unit circle.**

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Example 1: stable system

$$H(z) = \frac{1-0.9z^{-1}+0.18z^{-2}}{1-0.8z^{-1}+0.64z^{-2}}$$

$$z_1 = 0.3 \quad p_1 = 0.4000 + 0.6928j \quad |p_1| = 0.8 < 1$$

$$z_2 = 0.6 \quad p_2 = 0.4000 - 0.6928j \quad |p_2| = 0.8 < 1$$

Example 2: unstable system

$$H(z) = \frac{1-0.16z^{-2}}{1-1.1z^{-1}+1.21z^{-2}}$$

$$z_1 = 0.4 \quad p_1 = 0.5500 + 0.9526j \quad |p_1| = 1.1 > 1$$

$$z_2 = -0.4 \quad p_2 = 0.5500 - 0.9526j \quad |p_2| = 1.1 > 1$$

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1.4.5. LTI System Description. Summary

Time – Domain:

constant coefficient linear difference equation

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{M} a(k)y(n-k)$$

$h(n)$

Z – Domain:

transfer function

$$H(z) = \frac{\sum_{k=0}^{N} b(k)z^{-k}}{1 + \sum_{k=1}^{M} a(k)z^{-k}}$$

Z

Z^{-1}

FT

FT^{-1}

Frequency – Domain:

frequency response

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{N} b(k)e^{-j\omega k}}{1 + \sum_{k=1}^{M} a(k)e^{-j\omega k}}$$

$z = e^{j\omega}$ $\quad$ $e^{j\omega} = z$

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Time – Domain: impulse response $h(k)$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \qquad H(z) = \sum_{k=-\infty}^{\infty} h(k)z^{-k}$$

Z – Domain: transfer function $H(z)$

$$H\left(e^{j\omega}\right) = H(z)\Big|_{z=e^{jw}} \qquad h(n) = \frac{1}{2\pi j}\oint_C H(z)z^{n-1}dz$$

Frequency – Domain: frequency response $H\left(e^{j\omega}\right)$

$$H(z) = H\left(e^{j\omega}\right)\Big|_{e^{j\omega}=z} \qquad h(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega k}d\omega$$

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